A Twisting Electrovac Solution of Type II with the Cosmological Constant

J. K. Kowalczyński[1]

An exact solution of the current-free Einstein–Maxwell equations with the cosmological constant is presented. It is of Petrov type II, and its double principal null vector is geodesic, shear-free, expanding, and twisting. The solution contains five constants. Its electromagnetic field is non-null and aligned. The solution admits only one Killing vector and includes, as special cases, several known solutions.

KEY WORDS: Solutions of Petrov type II; electromagnetic field; cosmological constant

[1]Institute of Physics, Polish Academy of Sciences, Al. Lotników 32/46, 02-668 Warsaw, Poland. E-mail: jkowal@ifpan.edu.pl

This note presents an exact and explicit solution of the current-free Einstein–Maxwell equations with the cosmological constant. The solution in question may be written in the form

$$ds^2 = 2\left(r^2+n^2\right) d\zeta\, d\bar\zeta + 2\, dr\, k_\mu dx^\mu + W\left(k_\mu dx^\mu\right)^2$$

with the electromagnetic field tensor

$$\begin{aligned}
F_{\zeta\bar\zeta} &= \tfrac{1}{2}b\left(\zeta-\bar\zeta\right) + in\left\{\zeta\left[a-\tfrac{3}{2}b\left(r+in\right)^{-1}-iA\right]\right.\\
&\qquad \left.+\,\bar\zeta\left[a-\tfrac{3}{2}b\left(r-in\right)^{-1}+iA\right]\right\}\,,\\
F_{\zeta u} &= -a+\tfrac{1}{2}b\left(r+in\right)^{-1}+iA\,,\\
F_{\zeta r} &= in\bar\zeta F_{ur}\,,\\
F_{ur} &= \tfrac{1}{2}b\left[\zeta\left(r+in\right)^{-2}+\bar\zeta\left(r-in\right)^{-2}\right]\,,
\end{aligned}$$

where

$$k_\mu dx^\mu = du + in\left(\bar\zeta d\zeta - \zeta d\bar\zeta\right)\,,$$

$$\begin{aligned}
W &:= \left(r^2+n^2\right)^{-1}\left[\Lambda\left(\tfrac{1}{3}r^4+2n^2r^2-n^4\right)+2r\left(m+2ab\zeta\bar\zeta+Bu\right)-b^2\zeta\bar\zeta\right]\,,\\
A &:= \left(2n\right)^{-1}\left(b+C\right)\,,\quad B := n^{-2}b\left(b+C\right)\,,\quad C := \pm\left(b^2-4a^2n^2\right)^{1/2}\,,
\end{aligned}$$

and where ζ and $\bar\zeta$ are complex and conjugate coordinates, r and u are real coordinates, Λ is the cosmological constant, m is an arbitrary real constant, and a, b, and n are real constant arbitrary to a certain extent. Relations involving a, b, and n are discussed below.

Our solution is of Petrov type II iff $b \neq 0$. Its double Debever–Penrose vector is just k^μ determined by the 1-form $k_\mu dx^\mu$ given above, i.e. $k^\mu = \delta^\mu_r$. k^μ is geodesic and shear-free. The rates of expansion θ and of rotation ω of k^μ are given by the following complex equation:

$$\theta + i\omega = \left(r+in\right)^{-1}\,.$$

Thus, for every $r \neq 0$ we have $\theta \neq 0$, and $\omega = 0$ iff $n = 0$. k^μ is also a principal null vector of our electromagnetic field $\left(k_{[\mu}F_{\nu]\tau}k^\tau = 0\right)$, i.e. our case is aligned.

This field is non-null iff $b \neq 0$. Another Debever–Penrose vector (single if type II, double if type D; for subcases of Petrov type D see below), say l^μ, is determined by $l_\mu dx^\mu = dr + \frac{1}{2}Wk_\mu dx^\mu$. Our solution admits only one Killing vector, say ξ^μ, such that

$$\xi^\zeta = i\zeta\,, \qquad \xi^r = \xi^u = 0\,.$$

Our solution includes, as special cases, several known solutions. They can be obtained by eliminating some of the constants, without making infinite values of course. Note that A and B, and thus C, must be real.

If we put $a = b = 0$, then we eliminate the electromagnetic field and obtain the well-known luxonic variant (zero Gaussian curvature of a 2-space with the metric $(r^2+n^2)\,d\zeta\,d\bar{\zeta}$, $r =$ constant) of the Taub–NUT solution with the cosmological constant. This solution, found by many authors, is of Petrov type D iff $m \neq 0$ or $n\Lambda \neq 0$.

If we want to obtain subsolutions with the electromagnetic field but without the rotation ($n = 0$), then we have to assume that $a \neq 0$ or $b \neq 0$. If we put $b = 0$, then, according to our assumption, we have to keep $a \neq 0$. Then, however, A becomes imaginary, which is forbidden. (A occurs as an additive term in some of $F_{\mu\nu}$'s expressed in terms of only real coordinates, e.g. when $\zeta = x + iy$.) Thus we have to assume that $b \neq 0$ (but only at the beginning of the procedure, see below), and therefore we may not simply put $n = 0$ because of the negative powers of n in A and B. We may, however, consider the limiting transition $n \to 0$.

If $bC > 0$ (C being real of course), then the limiting transition $n \to 0$ is forbidden since it would make infinities.

If $bC < 0$ and $n \to 0$, then $A \to 0$, $B \to 2a^2$, and our ds^2 falls under a category of metric forms for which all the possible electromagnetic fields were found [1];[2] and then we obtain the solution (3.4) from [1], found earlier by Leroy [2].[3] In

[2]In [1] the signs of the cosmological constant (denoted therein by λ) are opposite to those commonly assumed, i.e. $\lambda = -\Lambda$.

[3]In [2] this solution is presented in a different coordinate system by eqs. (6.12c). It is quoted in the monograph [3] as eqs. (24.54d) where, in the second equation, x should read e^x

this solution, being of Petrov type II iff $b \neq 0$, a and b are independent. If we put $b = 0$, then we obtain a special case of some of the solutions listed in [1]. This special case ($b = n = 0$) is of Petrov type D iff $a \neq 0$ or $m \neq 0$, conformally flat iff $a = m = 0$ and $\Lambda \neq 0$, and flat iff $a = m = \Lambda = 0$.

If we assume that $C = 0$, then our solution is still of Petrov type II and twisting (iff $an \neq 0$, since $b^2 = 4a^2n^2$ in this case), but it contains only one electromagnetic constant, a, and does not contain the negative powers of n in A and B. If we put $n = 0$, then we obtain the special case described at the end of the preceding paragraph.

The solution presented in this note should be considered as new since, as far as I know, no solutions generalizing those listed in [1] (excluding solution (3.2) therein) have been published.

REFERENCES

1. Bajer, K., and Kowalczyński, J. K. (1985). *J. Math. Phys.* **26**, 1330.

2. Leroy, J. (1976). *Bull. Cl. Sci. Acad. R. Belg.* **62**, 259.

3. Kramer, D., Stephani, H., MacCallum, M. A. H., and Herlt, E. (1980). *Exact Solutions of Einstein's Field Equations* (VEB Deutscher Verlag der Wissenschaften, Berlin / Cambridge University Press, Cambridge).

4. Kramer, D., Stephani, H., MacCallum, M. A. H., and Herlt, E. (1984). "Exact Solutions of Einstein's Field Equations: Corrections." Preprint.

(multiplied by a proper constant; notation after [3]). This misprint is corrected on p. 11 in [4], but the correction is there unfortunately related to the next eqs. (24.54e) in [3].